A Chaotic Analysis of the Inertial Double Pendulum

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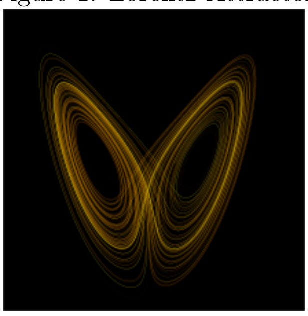
1 Abstract

This report examines the behavior of a classical chaotic system in the double pendulum, using numerical analysis and computer simulation. Using lagrangian mechanics, Lyapronuv exponents and further mathematical analysis the chaotic nature can be determined, which is then simulated through a simple java simulation. It is found that any minor change within the initial conditions will cause a significant difference to the simulation within a brief amount of time. The Lyapronuv exponents help us to show that there are both chaotic and non chaotic regimes for this system. Some of the issues associated with using a computer are also highlighted in this report as well, such as rounding errors in certain regimes. The dimensionless equations of motion are also derived and analysis of the critical points is undertaken.

2 Introduction

Chaos is an idea that is a fundamental component of the universe. It, in essence is the idea that some complex systems have a significant dependence on the initial point, or a sensitive dependence on the initial conditions. This means that a small change in the initial conditions will cause a huge change in the results further on in time. Most chaotic systems will never reach the exact same conditions in which they start within a finite amount of time. The classic example of a chaotic system is the double pendulum, which is what will be analyzed in this report. This is two pendulum of arbitrary mass and size connected to each other. Chaotic behavior is important to study as many real world systems, such as weather, traffic and behavior of some objects in space demonstrate many of its properties. The sensitivity to initial conditions means that it is extremely important to make sure that we know exact values. This idea is summarized well by Edward Lorenz, who states: "When the present determines the future, but the approximate present does not approximately determine the future." [1]. Chaotic Systems are an important part of nature, appearing at many times throughout astronomy and meteorology. Figure 1 displays a Lorenz attractor, which is a simple weather model. Whilst not appearing as chaotic as some other systems, a tiny difference in the initial conditions will cause the particle to end up at an entirely different spot on the diagram. Other examples

Figure 1: Lorentz Attractor



of applications of chaos theory appear all throughout science. Notably, population models developed by biologists can diverge exponentially. Most biological models are approximated as a continuous system, however some systems have been noted to exhibit chaotic growth models. In our system we will be examining two complex pendulum, (or pendulum with mass distributed along the arm), as shown in Figure 2. It is important to note that throughout this report the diagrams will appear to be simple pendulums, however this is simply because the simulation looks nicer that way. An example of what the figures that the simulation produces can be seen by Figure 3 In order to understand this system, we will use Lagrangian mechanics to develop our equations of motion. The full derivation can be seen in Appendix 1.

$$L = T - V = K_e - P_e \tag{1}$$

Figure 2: An inertial pendulum system

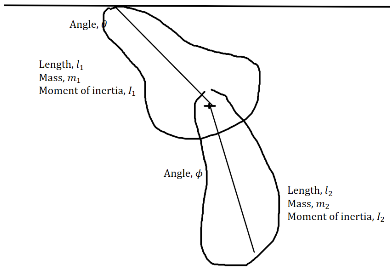
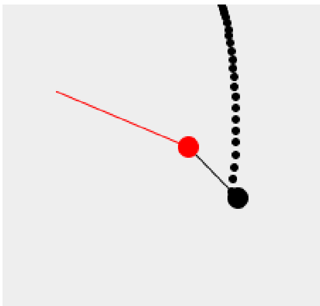


Figure 3: An example of how the pendulum simulation will like in this paper. It is important to note that, although they look like simple pendulum, they are actually complex pendulum, as seen by Figure 2



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{2}$$

By solving these, we end up with 3 and 4.

$$I_2\ddot{\phi} + m_2L_2^2\ddot{\phi} + m_2L_1L_2\ddot{\theta}\cos(\theta - \phi) - m_2L_1L_2\dot{\theta}^2\sin(\theta - \phi) + L_2m_2g\sin(\phi) = 0 \tag{3}$$

$$I_1\ddot{\theta} + m_2L_1L_2\ddot{\phi}\cos(\theta - \phi) + m_2L_1L_2\dot{\phi}^2\sin(\theta - \phi) + (m_1 + m_2)gL_1\sin(\theta) + (m_1 + m_2)L_1^2\ddot{\theta} = 0 \tag{4}$$

These equations are then used in a java simulation in order to gain an understanding of the physical nature of the system. Our simulations are run multiple times in order to gain exact results. Due to the nature of chaotic systems, and the way that they behave means that if there is a tiny change in variable, then we will end up with entirely different results later on in the experiment. A computer stores these values, but on occasion it can round the numbers slightly differently. This means that the after some finite time, we should not expect the exact same results, although across small time periods it is unlikely that these rounding errors could occur.

Due to this we can see effectively model and understand the nature of chaotic motion, and our simulation will allow us to visualize the experiments undertaken.

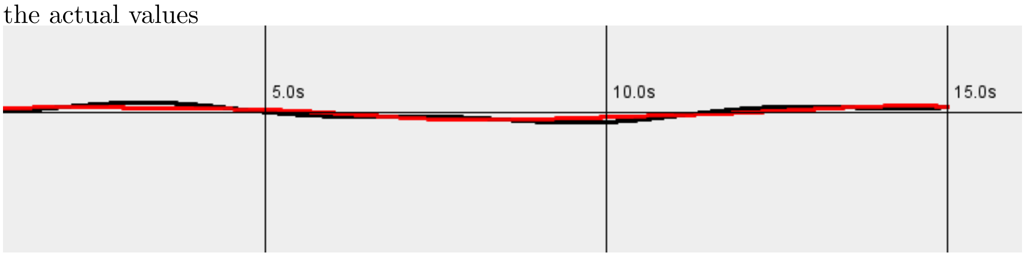
3 Stationary Points

A stationary point in our system are a set of initial conditions which will produce a 'non-moving' system. Or, more precisely, a state which will produce the same state when put into the equations of motion. This means that physically, they will never change from their original state. These can be derived numerically, by differentiating our equations of motion, and solving them. This solves to give a small set of stationary points, all of which occur at $\theta = 180$ or $\theta = 0$, and $\phi = 0$ or $\phi = 180$. Any combination of these points will result in a stationary point for the system

This produces our spectrum of stationary points, however it is also important to understand the exact behavior of these points, as some are considered stable, and some are unstable. A stable point is such that when there is a small change in the initial conditions, it will still keep the motion around the point where it started. An unstable point is such that if we change the initial position even slightly, it will end up chaotically diverging. These can be found mathematically by finding the derivatives very close to the stationary points nearby the original position. This can also be seen through our simulation, which has been run under all of the initial conditions for our system. The nature of our system is shown through the following:

- In the system where $\theta \approx 0$ and $\phi \approx 0$, we have our only stationary point that is stable. This is because even a small change will not drastically change the nature of the system. It will maintain a roughly similar position, as well as similar velocities. This can be seen through Figure 4, which demonstrates the roughly similar velocities. We can also see how it stays almost entirely in the low velocity zone that it starts in. As such, we can

Figure 4: The velocities of a pendulum with initial conditions close to 0 degrees. Vertical axis has been removed due to units being arbitrary. The shape of the graph is more important than the actual values



clearly see the nature of our stable critical point.

- The system where we have $\theta \approx 0$ and $\phi \approx 180$ is another stationary point, however this time it is an unstable scenario. This means that if we have an infinitesimal variation from our stationary point the entire system will change from it's initial coordinates, and won't reach the stationary point again. As such, it has an entirely different plot to the one seen in Figure 4, diverging chaotically. We can see this divergence from the initial conditions through Figure 5.

Figure 5: The chaotic divergence of a system with initial conditions of close to 0 and 180

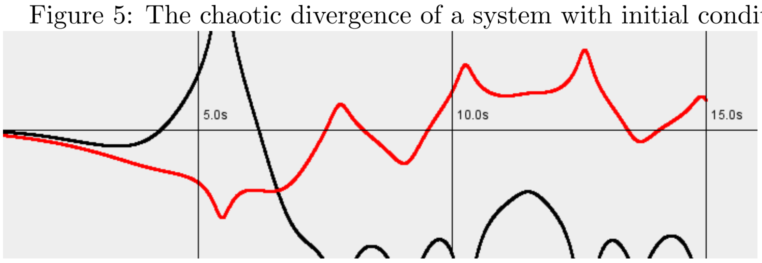


Figure 5 shows us how the system diverges almost immediately after the starting position.

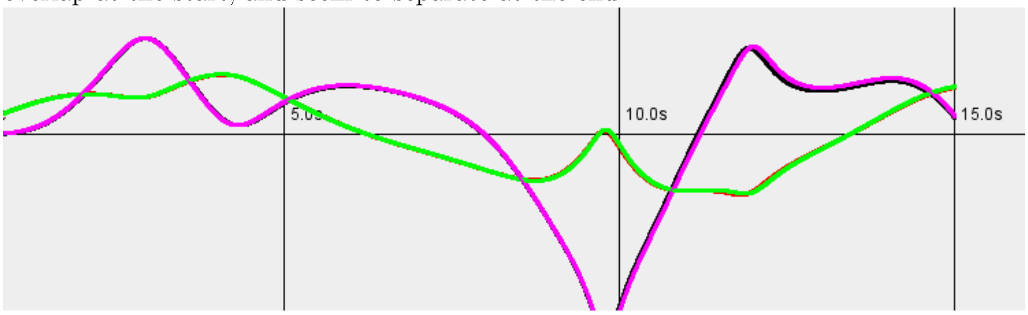
The velocity starts out close to 0, but very soon after evolves drastically. Comparing this to our stable point seen in 4, we can see the significant difference of the stable and the non stable points, and how they evolve with time. We could further analyse the other critical points, however it will simply produce similar chaotically diverging graphs. This is because every point apart from the one seen in 4 is unstable

From this we can see the different types of stationary points and how they evolve with time. We can also understand the difference between a stable and an unstable point, and how, in our system, that even very close to a stationary point we can have chaotic divergence if it is an unstable regime.

4 Chaotic Divergence

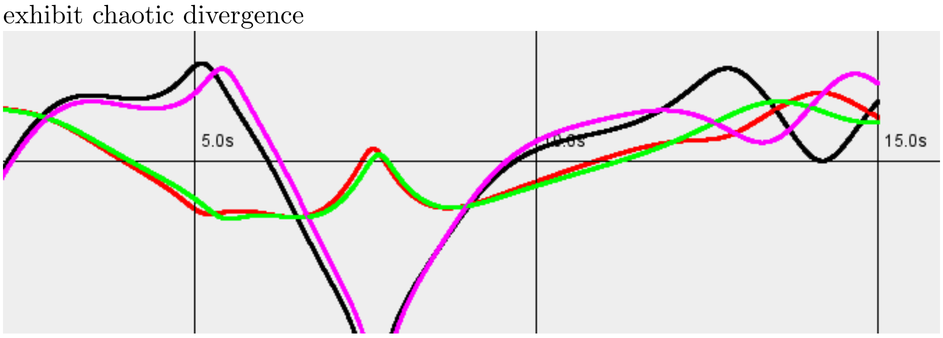
A criteria for a system to be chaotic is that if we have two initial points that are very close, that they will eventually diverge. In order to show this, we can run our simulation over several iterations with similar initial conditions. To show this experimentally, we will run our simulation with two sets of pendulums, with two sets of initial conditions displaced by a tiny amount. Our first system of pendulums, (which will be represented by the red and black graphs, where red is the first arm and black is the second) will be run simultaneously (represented by the green and purple graph) as our second system. The second system will be displaced by one degree, in order to demonstrate how similar sets of initial conditions can diverge. Figure 6

Figure 6: This demonstrates the first 15 seconds of our simulation. Note that the two graphs overlap at the start, and seem to separate at the end



displays the first 15 seconds of our divergence test. The two start out very similar, however we can see some of the divergence towards the end of the graph. They are indistinguishable as the Purple and green plots are overlapping. 7 demonstrates the further chaotic divergence of our

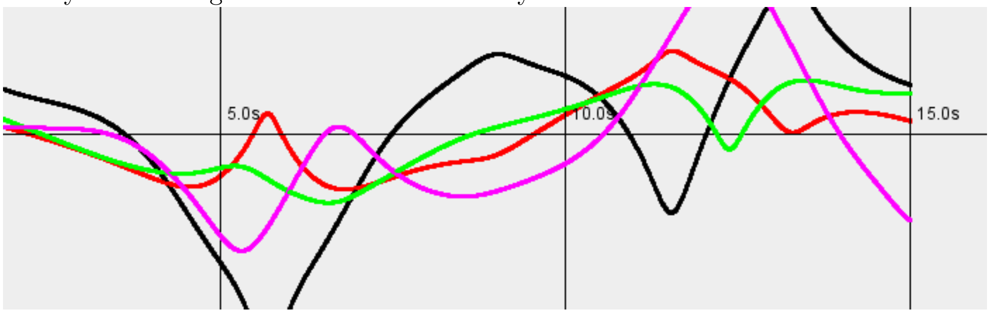
Figure 7: This graph demonstrates the next 15 seconds of our pendulums running simultaneously. We can see the divergence at the end of the figure, as the two pendulums begin to exhibit chaotic divergence



system. This clearly shows that the system is evolving into two completely different systems. We can further see this through 8 which shows the final 15 seconds of our system, where it truly does become chaotic, with the different plots each diverging into chaos. This suggests that they will have entirely different velocities, meaning that neither will ever be as similar as they were at the start. Hence we can see the chaotic divergence of systems as close as 1 degree, hence demonstrating the chaos that is in this system. As such our systems sensitivity to initial conditions is demonstrated.

If this is repeated with almost any initial conditions of pendulum with any variable changed by any amount we will find that the two will always diverge away from each other. This even happens in the very small regions, such as the low energy regime. If however, we have our second pendulum left massless and rigid (infinite moment of inertia) our system effectively devolves into a single pendulum with a bizarrely shaped arm. This system will not exhibit the same chaotic divergence that we see in this section as it loses the characteristics of a double

Figure 8: This demonstrates the last 15 seconds of our system, where we can clearly see the two systems diverge into their own chaotic systems



pendulum, while still technically being a double pendulum. As such, we do not get the chaotic divergence through every single set of initial conditions. It will exhibit chaotic divergence for all reasonable representations of our double pendulum that do not remove the behavior of the pendulum.

5 Dimensionless Equations

A dimensionless equation is one such that there is the removal of some or all of the physical units from the initial equation. It allows many problems with measured units to become much simpler, and is regularly used in dimensional analysis. It also can recover all of the properties of the system, and makes the calculation and physical analysis become simpler. "There are three important motivations for writing complex equations in dimensionless or dimensionally reduced form.

1. It is easier to recognize when to apply familiar mathermatical techniques
2. It reduces the number of times we might have to solve the equation numerically.
3. It gives us insight into what might be small parameters that could be ignored or treated approximately." 3

By examining our system given by 3 and 4 we find that we have many physical parameters. We can hence simplify this equation by converting it into its dimensionless equivalent.

$$\tau = \frac{t}{t_0} \tag{5}$$

This is done through the converting time into a dimensionless constant, denoted by 5. So, by making this substitution and simplifying, we end up with 6 and 7. The full derivation of these equations can be seen in Abstract 2.

$$\frac{\partial^2\phi}{\partial\tau^2}\cos(\theta-\phi)+(\frac{\partial\phi}{\partial\tau})^2\sin(\theta-\phi)+\frac{(I_1+(m_1+m_2)L_1^2)}{m_2L_1L_2}\frac{\partial^2\theta}{\partial\tau^2}+\frac{L_1(m_1+m_2)}{m_2L_2}\sin(\theta)=0 \tag{6}$$

$$\frac{\partial^2\theta}{\partial\tau^2}\cos(\theta-\phi)-(\frac{\partial\theta}{\partial\tau})^2 sin(\theta-\phi)+\sin(\phi)+\frac{(I_2+m_2L_2^2}{m_2L_1L_2}\frac{\partial^2\phi}{\partial\tau^2}=0 \tag{7}$$

As such we have derived our dimensionless equations, and we will use them to further simplify our calculations and simplify our analysis later on.

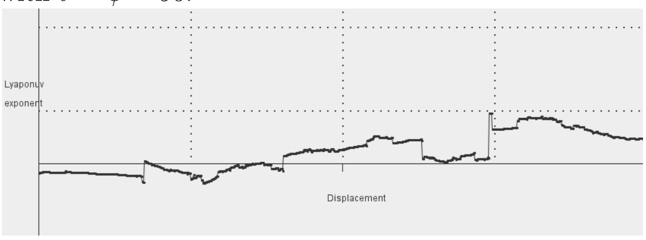
6 Lyapunov Exponents

The Lyapunov exponent is a method of quantifying the chaos inside a system [4].This is rigorously defined to be 8[2]. In this equation, d_0 is the initial displacement with our initial conditions and $d(t)$ is the distance between our solutions, after some time, t has passed. The higher and more positive this value is, the more chaotic the system is defined to be. If this value is negative, our system is non chaotic. It is directly related to the exponential divergence of nearby trajectories. However, this is quite difficult to calculate, particularly with our complicated system. As such, we will calculate successive components over a finite time interval, and determine the average Lyapunov exponent over a spectrum of initial conditions.

$$\lambda = \lim_{t\to\infty} \lim_{d_0\to\infty} \frac{1}{t} \ln(\frac{d(t)}{d_0}) \tag{8}$$

This will give us a good approximation for our lyapunov exponents, instead of a very difficult calculation like above. The average value of many successive Lyapunov spectrums will provide a value which converges to the value that would be found from 8. From plotting this spectrum we end up with Figure 9, which shows a plot of our spectrum. It seems that the exponent calculated is random, this is however expected for our lyapunov exponent. When we take the

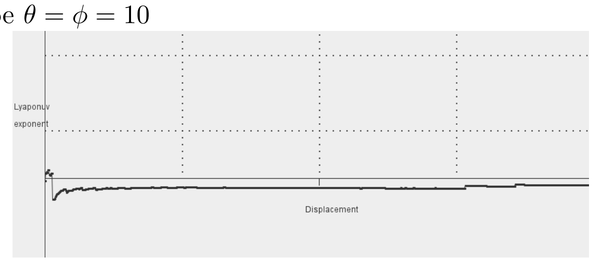
Figure 9: This demonstrates our lyapunov spectrum graph with initial conditions seperated by 0.0001 radians, with $\theta = \phi = 90$.



average of these values over this period we find that our average exponent $\bar{\lambda}$ is equal to 0.163. This indicates the chaotic nature of our system, as a positive Lyapunov exponent indicates chaos. If we take individual components, we find that some are negative which implies a non chaotic regime. This is why we need to take a spectrum of our exponents so that our value will converge to the actual result. As such we can see that the system denoted by 9 is a chaotic regime.

An example of the non-chaotic case is when we change our angles from 90, to 10. This is shown through Figure 10, which has a lyapunov exponent indicating non-chaotic motion, converging to a value of -0.959. A comparison of these two systems clearly demonstrates the difference between the two of them. The chaotic system has a nearly impossible curve to predict, whereas the non chaotic regime almost immediately converges to it's negative value.

Figure 10: This figure demonstrates a non chaotic regime for the lyapunov exponents, with initial angles set to be $\theta = \phi = 10$



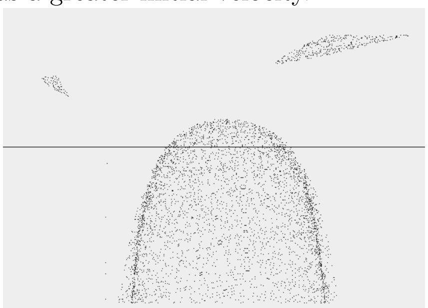
7 Poincaré Sections

A Poincaré section or Poincaré map is when a periodic orbit, with some initial conditions, which plots the positions which will return to this section. It then forms a map of points which will send the first point to the second [5]. It is often used for analyzing the original system in a much simpler manner. In practice however, it is not always possible as there is no general method to construct a Poincaré map.

Our system, having so many parameters is incredibly difficult to visualize, which is why using poincaré maps make it much easier for us to understand the nature and behavior of our system.

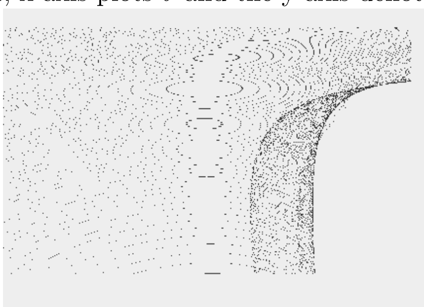
In order to do this we will need to single out one, or several of our dimensions, which allows us to visualize it in a 2 dimensional 'slice'. In this system, we will freeze out ϕ and $\dot{\phi}$. They are set to be 0, and then whenever our system reaches $\theta = 0$ in some given time frame, we plot the position on a velocity position graph. This allows us to easier visualize the behavior of our system, as we can get see how it evolves around a certain point. We can get a spectrum of Poincaré spectrum for a variety of initial conditions, allowing us to understand the variety and behavior of our system.

Figure 11: This image demonstrates a simple poincare map. On the x axis we have θ, and along the y axis we have $\dot{\theta}$ We can see the behavior of the system as it more likely to reach 0 degrees in a system which has a greater initial velocity.



We can see some stable elliptical orbits for our system through figure 12, which has several regions of stability seen through the loops in the image. Both figures 11 and 12 demonstrate

Figure 12: Poincaré section of the system with $\theta = \phi = 90^o$ and the masses of the rods being 10 units. Note how there are several elliptical zones of instability through the center of the image. Similarly to figure 11, x axis plots θ and the y axis denotes $\dot{\theta}$.

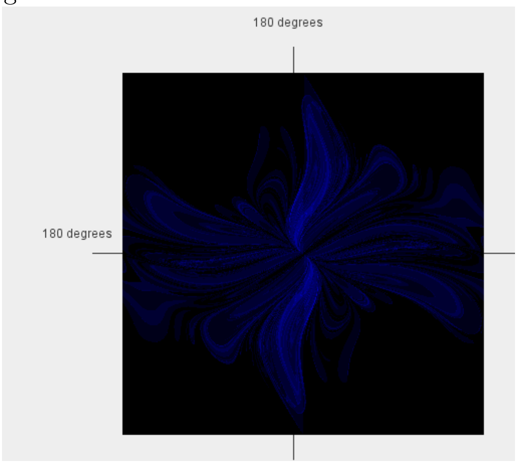


chaotic motion as seen by the almost random scattering of points through the system. We do however see some non chaotic behavior through the elliptical nature in figure 12. As such, we can see some chaotic and non chaotic regimes of our system.

8 Flip Analysis

We can determine other interesting information about the double pendulum by determining the time it takes for one, or either of the pendulum to flip over. This is done by determining when the either arm changes its angle to either side of 180 degrees. Figure 13 demonstrates

Figure 13: A map of the amount of flips performed by the system in a given time, with our pendulum with equal length arms



this map. This image is a 360×360 image, where each pixel represents the coordinate of the arms, so for example, the (1, 1) pixel is the point where our pendulum has $\theta = 1$ and $\phi = 1$. The colour of the pixel is brighter depending on the amount of flips that the system performs in a given time frame. So the more flips it does, the closer it gets to a 'white' pixel. In this image it is difficult to see the chaotic nature of the pendulum, so in figure 14, we have a system where the mass of the second arm is half that of the upper arm, and the length of the lower arm is shorter. This is to encourage flipping of the system. In this case, we can clearly see that there are more flips throughout the system. This is because it is a system which is designed to encourage the flipping behavior. There is also flipping across a much wider range of in this system, as we can see the 'flips' across almost the entire image.

It is also important to note, that this particular section of code is particularly computationally expensive, taking several minutes in order to produce the image. There is also a unique

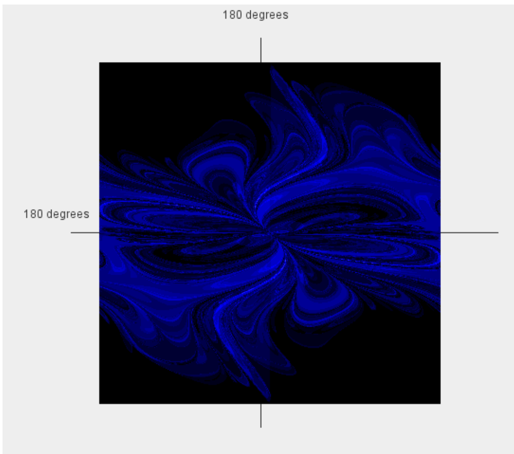
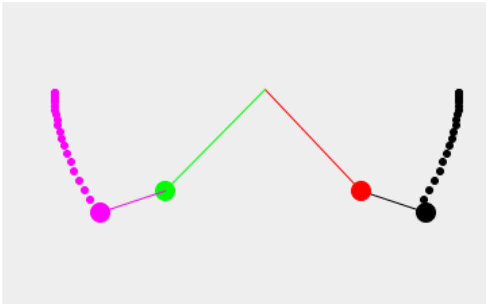


Figure 14: This figure denotes another flip map, this time with the second arm with half the mass, and the second arm being two fifths as long as the upper arm.

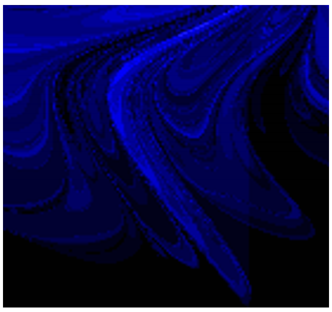
Figure 15: This demonstrates the symmetry of a system that starts with$\theta = \phi = 90^o$ and $\theta = \phi = 270^o$. It becomes obvious that they will travel symmetric paths



kind of reflected symmetry in these images. This might seem to violate the principle of chaotic motion (that any change in the initial conditions will lead to different results). However, when we consider that in this map we will have both the systems $\theta = \phi = 90^o$ and $\theta = \phi = 270^o$. These systems are exactly the same, just reflected around the center, and according to Galileo [6], all experiments should be the same, regardless of their reference frames. This Makes sense, because both of our systems are exactly the same, just being on opposite sides. As such, it makes perfect sense that our image has this line of symmetry. This is further demonstrated by Figure 15, which shows that the two pendulum are the same system reflected around the center axis.

As we can see from this image, they are the same system just reflected, so naturally they will have the same amount of flips in a given time frame. As such, the symmetry seen in Figures 13 and 14 can be explained by this symmetry. Figure 14 also depicts some fractal nature, as

Figure 16: This image is a zoomed in portion of figure 14, so we can closely analyze the fractal edge of the image

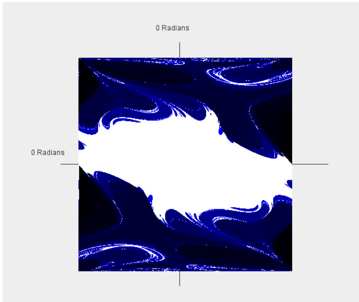


if we pick an 'edge' of one of the images, as shown in Figure 16 we can clearly see that it has no defined edges between the chaotic and non chaotic portions of the map. It is possible to increase the resolution of the image, by increasing the amount of iterations that the simulation completes. This however will not do anything more than add more lines, creating more edges. This means that it exhibits the properties of a fractal image. It is important to note however, that there will be some regions where there will be no flips at all, irrespective of how fine we make our resolution. These regions are such that there is not enough combined energy (kinetic and potential) to get the lower arm above 180 degrees. These regions exist around the edges of the image (in the 0,0 area), and depend on all of the physical parameters. Hence we can understand why we do not see much fractal nature around the edges of the image.

The fractal nature of an image is defined as a never ending patter that repeats itself in different scales [7]. Our image, although non repeating exhibits some of these properties, primarily because if we attempt to determine the edge of one of our sections, we will find it becomes infinitely small. Hence, our flip maps demonstrate some of fractal nature. Figure 16 also demonstrates the inherint chaos inside a system. If we transition within 10 degrees on the image in any direction, we find tha twe can go from many flips to 0. This demonstrates that the behaviour of the system is largely very difficult to predict and to anticipate in a finite time span.

We can also form a plot of how long it takes for the system to produce a single flip. This demonstrates in a similar sense the chaos of the system, as we can clearly see that very similar trajectories will have entirely different times to produce a flip. This can be see through figure 17, which is produced in a similar manner to the previous images in this section. Every pixel in the image represents a coordinate, however in this case we go from $-\pi$ to π

Figure 17: This plot demonstrates the time taken for the pendulum to flip over. The white occurs if the simulation doesn't flip, and the brighter the blue the sooner the image flips.

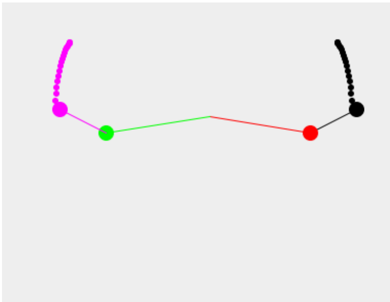


This image appears to be exactly as we expect it, as in the center (where $\theta = \phi = 0$) we find that there is no flipping, and as we get towards the edges of the image there is a much shorter time for the simulation to flip. We also can see the stationary point where $\theta = 0$, as we have no flipping along the middle line of the image. This too demonstrates the behavior of a fractal image, except much more rigorously than figure 14. We can prove this through the box method of fractals, however due to limitations in computational capacity, this is impossible to show in great detail. If this method were to be undertaken, it would involve 'zooming' in on a particular area of the image, and demonstrating that no matter how far we go in, it is impossible to succinctly define an edge.

9 Problems With using a computer

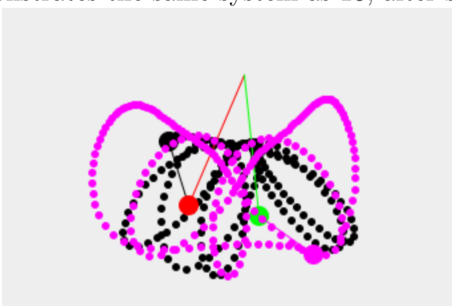
Due to the nature of using a computer to simulate our system we can come across some problems. A computer will on occasion store a variable to different rounding in order to save space, or it can be updated slightly differently [8]. Due to a chaotic systems sensitivity to initial conditions, if there is any change in our systems position, eventually the two systems, which were initially identical, will diverge. This is best demonstrated through figures 18 and 19, which show the same simulation at different time frames. Figure 18 demonstrates the initial conditions of the system. According to [6] these systems should behave exactly the same, just reflected around the centre axis. However, after running the simulation for some time ($\approx$ 180 seconds on my computer) the two systems diverged into the system given by Figure 18

Figure 18: This system is the simulation ran at $\theta = \phi = 120$ for the first pendulum set, and our second set of pendulums at $\omega = \eta = 240$. These systems should be equivelant, just reflected about the center axis



demonstrates the initial conditions of the system. According to [6] these systems should behave exactly the same, just reflected around the centre axis. However, after running the simulation for some time ($\approx$ 180 on my computer) the two systems diverged into the system given by 19 This can be seen through any combination of initial conditions that represent the same system,

Figure 19: This demonstrates the same system as 18, after some time has passed.



however the divergence due to computer error seems to occur sooner at systems with a greater energy. As such, we must consider any results in systems with large time scales less accurate than shorter experiments. This can be solved with an infinite precision computer, which is physically impossible. Another alternative is to write a custom memory storage program, however for the purposes of our experiment, it is unnecessary for a huge amount of precision over long time frames.

10 Conclusion

Our system is demonstrated to be chaotic, both through computational experimentation and mathematical demonstration with our Lyapunov exponents. We can also understand the behavior and nature of our pendulum through our poincaré maps, which show how certain regimes can have attractors around certain points and other regimes can appear to have a completely arbitrary distribution.

We also demonstrated some of the limits in using a computer to simulate our scenario. We demonstrate how small rounding errors made by computers will eventually propagate, causing simulations which are exactly the same to eventually diverge and separate. In reality if we could replicate exact conditions, this shouldn't happen however error rounding, and the sensitivity to initial conditions can cause our system to diverge.

As such, a clear understanding on the behavior and characteristics of the inertial double pendulum can be clearly understand and modeled. We can see the chaotic divergence of most regimes. We can also see the stationary points of our system, and the chaotic distribution of 'flipping' as well as the time taken for these 'flips' to occur.

11 Appendix

11.1 Appendix 1: A derivation of the equations of motion

In order to derive the equations of motion we will need to find the Kinetic and potential energies of our system in order to satisfy 10.

$$L = T - V = K_e - P_e \tag{9}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0 \tag{10}$$

We will begin by converting our coordinates from x and y into θ and ϕ, which can be seen seen in 11 through 20. The following are the coordinate conversions for the first pendulum.

$$L_1^2 = x_1^2 + y_1^2 \tag{11}$$

$$x_1 = L_1 \sin(\theta) \tag{12}$$

$$\dot{x_1} = L_1 \dot{\theta} \cos(\theta) \tag{13}$$

$$y_1 = -L_1 \cos(\theta) \tag{14}$$

$$\dot{y_1} = L_1 \sin(\theta)\dot{\theta} \tag{15}$$

We will now derive the equations for our second pendulum, as seen below:

$$L_2^2 = x_2^2 + y_2^2 \tag{16}$$

$$x_2 = L_1 \sin(\theta) + L_2 \sin(\phi) \tag{17}$$

$$y_2 = -L_1 \cos(\theta) - L_2 cos(\phi) \tag{18}$$

$$\dot{x_2} = L_1 \cos(\theta)\dot{\theta} + L_2 cos(\phi)\dot{\phi} \tag{19}$$

$$\dot{y_2} = L_1 \sin(\theta)\dot{\theta} + L_2 \sin(\phi)\dot{\phi} \tag{20}$$

These can then be used to determine the kinetic and the potential energies of the system, which when substituted into our lagrangian we get the following 21

$$L = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}^2 + \frac{1}{2}\dot{\theta}^2 I_1 + \frac{1}{2}\dot{\phi}^2 I_2 + m_2 L_1 L_2 \dot{\theta}\dot{\phi} cos(\theta-\phi) + \frac{1}{2} m_2 L_2^2 \dot{\phi}^2 + (m_1+m_2) g L_1 \cos(\theta) + m_2 g L_2 \cos(\phi) \tag{21}$$

Thus demonstrating our derivation of our Lagrangian. We will then use this to substitute into 10, and then to find our equations of motion. This is done using mathematica, to ensure our equations are correct. After having done this we get our two equations of motion, given by 22 and 23.

$$I_2\ddot{\phi} + m_2 L_2^2 \ddot{\phi} + m_2 L_1 L_2 \ddot{\theta} \cos(\theta - \phi) - m_2 L_1 L_2 \dot{\theta}^2 \sin(\theta - \phi) + L_2 m_2 g \sin(\phi) = 0 \tag{22}$$

$$I_1\ddot{\theta} + m_2 L_1 L_2 \ddot{\phi} \cos(\theta-\phi) + m_2 L_1 L_2 \dot{\phi}^2 \sin(\theta-\phi) + (m_1+m_2) g L_1 \sin(\theta) + (m_1+m_2) L_1^2 \ddot{\theta} = 0 \tag{23}$$

Hence we can see the derivation of our equations of motion for this system.

11.2 Appendix 2: A derivation of the dimensionless equations

This section aims to derive our dimensionless equations. In order to do this, we will make the substitution 24 into 22. Doing this and simplifying will result in 25, which has been simplified using the chain rule.

$$t_0 = \sqrt{\frac{L_1}{g}} \tag{24}$$

$$(m_1+m_2)gL_1\sin(\theta)+\frac{\partial^2\theta}{\partial\tau^2}\frac{(I_1+(m_1+m_2)L_1^2)}{(t_0^2)}+\frac{m_2L_1L_2}{t_0^2}(\frac{\partial\phi}{\partial\tau})^2\sin(\theta-\phi)+\frac{m_2L_1L_2}{\tau_0^2}\frac{\partial^2\phi}{\partial\tau^2}\cos(\theta-\phi)=0 \tag{25}$$

By substitution into 23 we can get our other dimensionless equation, as seen through 26.

$$\frac{I_2+m_2L_2^2}{t_0^2}\frac{\partial^2\phi}{\partial\tau^2}+m_2gL_2\sin(\phi)+\frac{m_2L_1L_2}{t_0^2}\frac{\partial^2\theta}{\partial\tau^2}\cos(\theta-\phi)-\frac{m_2L_1L_2}{t_0^2}(\frac{\partial\theta}{\partial\tau})^2\sin(\theta-\phi)=0 \tag{26}$$

This is then simplified with mathematica, which then gives us our final dimensionless equations, as seen through 27 and 28

$$\frac{\partial^2\phi}{\partial\tau^2}\cos(\theta-\phi)+(\frac{\partial\phi}{\partial\tau})^2\sin(\theta-\phi)+\frac{(I_1+(m_1+m_2)L_1^2)}{m_2L_1L_2}\frac{\partial^2\theta}{\partial\tau^2}+\frac{L_1(m_1+m_2)}{m_2L_2}\sin(\theta)=0 \tag{27}$$

$$\frac{\partial^2\theta}{\partial\tau^2}\cos(\theta-\phi)-(\frac{\partial\theta}{\partial\tau})^2 sin(\theta-\phi)+\sin(\phi)+\frac{(I_2+m_2L_2^2}{m_2L_1L_2}\frac{\partial^2\phi}{\partial\tau^2}=0 \tag{28}$$

11.3 Appendix 3: Code Used to build this assignment

Link to code which was used to generate the pictures, graphs and other stuff within this assignment

12 References

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